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The Weyl bundle as a differentiable manifold

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Abstract
The construction of an infinite-dimensional differentiable manifold $\mathbb{R}^\infty$ not modelled on any Banach space is proposed. Definition, metric and differential structures of a Weyl algebra $(P_\mathrm{p}^* M[[\hbar]], \circ)$ and a Weyl algebra bundle $(\mathcal{P}^*\mathcal{M}[[\hbar]], \circ)$ are presented. Continuity of the $\circ$-product in the Tichonov topology is proved. Construction of the $*$-product of the Fedosov type in terms of theory of connection in a fibre bundle is explained.

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1. Introduction

Deformation quantization was born twice. First the complete version of quantum mechanics in the language of classical physics appeared in the middle of the previous century, when Moyal [1] using works by Weyl [2], Wigner [3] and Groenewold [4] presented quantum mechanics as a statistical theory. His results were only valid for the case of R^{2n}.

For the second time deformation quantization appeared 30 years later. Ever since two papers by Bayen *et al* [5] were published in 1978, a great interest in that version of quantum mechanics has been observed.

One possible realizations of the deformation quantization programme is the so-called Fedosov formalism [6, 7]. In its original version the Fedosov approach to quantum mechanics, although based on the theory of connection in a bundle, is a purely algebraic construction. A 1-form of connection or a 2-form of curvature appear as objects belonging to some algebra bundle and acting on elements from that bundle via commutators.

The presence of connection and curvature in Fedosov's formalism framework suggests that this topic may be treated in terms of differential geometry. As Fedosov machinery is a part of quantum theory, its geometrization is a geometrization of deformation quantization. Profits from such a treatment of a physical theory are obvious: one obtains a clear definition of continuity, it is easy to represent the derivation in a form covariant under some transformations etc. Since then we find it worth reformulating the Fedosov formalism in more geometrical language. The current paper is one of two papers (see [8]) devoted to this problem.

The main role in the Fedosov version of deformation quantization is played by a Weyl bundle which is an infinite-dimensional differentiable manifold. Infinite-dimensional differentiable manifolds have appeared in the development of physics several times. An example of such a manifold is the Hilbert space of a quantum harmonic oscillator. But all known infinite-dimensional differentiable manifolds *are modelled on some Banach spaces* [9]. And the Weyl bundle is not normalizable. To avoid this fundamental obstacle we propose a new look at an infinite-dimensional manifold and explain its deep physical origin. Our considerations are related not only to deformation quantization. They can also be useful in the theory of self-dual Yang–Mills (SDYM) equations for the $*$-bracket Lie algebra (see [10, 11]).

In the second section we analyse a space of infinite real series $\mathbb{R}^\infty$. We equip it in a metric and a topological structure showing that it is a Hausdorff space and also a Fréchet space. After that we define an atlas on that space of infinite real series and explain how to extend it to a complete atlas so we prove that $\mathbb{R}^\infty$ is a differentiable manifold. The space $\mathbb{R}^\infty$ appears as a dual space to the vector space of polynomials with real coefficients.

The next part of our contribution is devoted to a Weyl algebra $(P^*_{\mathrm{p}}M[[\hbar]], \circ)$. We show that this space is a metrizable complete space modelled on the differentiable manifold $\mathbb{R}^\infty$ introduced in the second section. At the end of this part we prove that the $\circ$-product is continuous in the Tichonov topology.

The fourth section deals with the construction of a Weyl algebra bundle $\mathcal{P}^*\mathcal{M}[[\hbar]]$. We show that the collection $\bigcup_{\mathrm{p}\in M} P^*M_{\mathrm{p}}[[\hbar]]$ of the Weyl algebras is really a differentiable manifold and, moreover, is a vector bundle.

The fifth part of our contribution is devoted to the construction of connection in the Weyl bundle. We start from a brief review of the theory of connection in vector bundles and after that we introduce symplectic connection in $\mathcal{P}^*\mathcal{M}[[\hbar]]$. In the last part of this paper, using that symplectic connection, we propose an Abelian connection in $P^*_{\mathrm{p}}M[[\hbar]]$ and explain its role in deformation quantization. In contrast to Fedosov we introduce both connections in terms of differential geometry and show that the algebraic method proposed by Fedosov is a special case (working only in Darboux atlases) of our more general treatment. Moreover, we prove that the symplectic connection is the only induced connection on the Weyl bundle $\mathcal{P}^*\mathcal{M}[[\hbar]]$ which can be expressed by the $\circ$-product.

In the appendix the proof of the relation between the position of an element of the Weyl algebra and its indices is presented.

As this text concentrates on the differential aspect of the Weyl bundle we do not consider its algebraic properties. Formal construction of the Weyl bundle as a bundle of elementary C^* algebras is contained in [12].

Our paper was written by a physicist for physicists. This is the reason why we decided to quote a lot of definitions and theorems which are well known to mathematicians. Another reason is that even such fundamental ideas as a differentiable manifold or curvature are defined in slightly different ways by different authors.

The bibliography of the Fedosov formalism and its applications is rather wide. Hence we mention only works [13–19] which represent the geometrical trend in this subject.

The reader who finds the compendium of topology or differential geometry presented in our paper unsatisfactory is recommended to look into [20–24]. Parts devoted to theory of fibre bundles are based on [27, 28].

Finally a short comment about notation is needed. We use the Einstein summation convention but in all formulae in which we find it necessary we put also the symbol Σ. For example in the fifth section when in the same expression we have two or more sums in different intervals we use symbols of summation.

2. $\mathbb{R}^\infty$ as a differentiable manifold

In this section we introduce an infinite-dimensional differentiable manifold which is the natural generalization of a space $\mathbb{R}^n$, $n \in \mathcal{N}$. Such an object will be required to analyse differential properties of the Weyl bundle.

Let us start from the one-dimensional (1D) case. A pair $(\mathbb{R}, \varrho)$ is a metric space with the distance defined as

$$\varrho(x, y) \stackrel{\text{def.}}{=} |x - y| \qquad \text{for each} \quad x, y \in \mathbb{R}. \tag{2.1}$$

A set of balls

$$K(c, r) = \{x \in \mathbb{R}, |x - c| < r\}, \qquad c \in \mathbb{R}, \quad r > 0,$$

determines the topology $\mathcal{T}$ on $\mathbb{R}$ so $(\mathbb{R}, \mathcal{T})$ is a topological space.

Let us construct a space $\mathbb{R}^\infty$ as the infinite Cartesian product $\Pi_{i=1}^{\infty}\mathbb{R}_i$, where $\forall_i \mathbb{R}_i = \mathbb{R}$. In the space $\mathbb{R}^\infty$ the topology $\Pi_{i=1}^{\infty}\mathcal{T}_i$ is defined as follows.

Definition 2.1. *[20] For each $x = (x^1, x^2, \ldots) \in \mathbb{R}^\infty$ the topological basis of neighbourhoods of the point x is all sets $\mathcal{U} = \Pi_{i=1}^{\infty}\mathcal{W}_i$, where*

$$\mathcal{W}_i = \begin{cases} \mathbb{R} \text{ for each } i \text{ apart from the finite number of indices,} \\ \mathcal{U}_j(x^j) \text{ for the rest of indices.} \end{cases}$$

By $\mathcal{U}_j(x^j)$ we mean an arbitrary 1D neighbourhood of $x^j \in \mathbb{R}$. The mapping

$$P_i : \mathbb{R}^\infty \to \mathbb{R}_i$$

such that $P_i(x) = x^i$ is called a projection *of $\mathbb{R}^\infty$ on $\mathbb{R}_i$. The topology $\Pi_{i=1}^{\infty}\mathcal{T}_i$ is known as the* Tichonov topology.

The Tichonov topology is the coarsest topology in which the projections P_i are continuous.

The Tichonov topology can also be introduced by a set of seminorms. The advantage of this method is that $\mathbb{R}^\infty$ becomes in a natural way a Fréchet space. So to investigate its properties we are able to use the powerful machinery of the theory of Fréchet spaces.

Definition 2.2. *[22] Let V be a vector space over $\mathbb{R}$. A mapping $[|\cdot|] : V \to \mathbb{R}$ is called a* seminorm *if*

1. $\forall_{x \in V} [|x|] \geqslant 0$;
2. $\forall_{x \in V} \forall_{t \in \mathbb{R}} [|t \cdot x|] = |t| \cdot [|x|]$;
3. $\forall_{x,y \in V} [|x + y|] \leqslant [|x|] + [|y|]$.

An open ball in a seminorm $[|\cdot|]$ is a set of points such that

$$K(c, r) = \{x \in V, [|x - c|] < r\}, \qquad c \in V, \quad r > 0. \tag{2.2}$$

Let $[|\cdot|]_i$ be a seminorm in a vector space $\mathbb{R}^\infty$ defined by

$$[|\cdot|]_i : \mathbb{R}^\infty \to \mathbb{R}_i, \qquad [|x|]_i \stackrel{\text{def.}}{=} |x^i|. \tag{2.3}$$

A set of seminorms $\{[|\cdot|]_i\}_{i \in \mathcal{N}}$ in $\mathbb{R}^\infty$ determines a topology on $\mathbb{R}^\infty$ in the following way.

A set $\mathcal{U} \in \mathbb{R}^\infty$ is open in the topology compatible with seminorms $\{[|\cdot|]_i\}_{i \in \mathcal{N}}$ if for every $x \in \mathcal{U}$ there exists $i_1, \ldots, i_r \in \mathcal{N}$ and $\epsilon > 0$ such that

$$K_{i_1}(x, \epsilon) \cap K_{i_2}(x, \epsilon) \cap \cdots \cap K_{i_r}(x, \epsilon) \subset \mathcal{U}$$

i.e. each point of U belongs to an intersection of a finite number of balls contained in U. By $K_{i_l}(x, \epsilon)$ we denote an open ball (2.2) in the seminorm $[|\cdot|]_{i_l}$.

The topology determined by the set of seminorms $\{[|\cdot|]_i\}_{i\in\mathcal{N}}$ is the same as the Tichonov topology in $\mathbb{R}^\infty$ introduced before (see [20]).

Theorem 2.1. *[22] A vector space V with topology $\mathcal{T}$ defined by the set of seminorms $\{[|\cdot|]_z\}_{z\in J}$ (J is countable or not) is the topological vector space $(V, \mathcal{T})$. It is also a Hausdorff space iff the neutral element $\Theta \in V$ is the only vector such that $\forall_{z\in J}[|\Theta|]_z = 0$.*

Using the above theorem we conclude that the pair $\left(\mathbb{R}^\infty, \Pi_{i=1}^\infty \mathcal{T}_i\right)$ is a Hausdorff space.

Definition 2.3. *[22] A Hausdorff topological vector space $(V, \mathcal{T})$ is called* pre-Fréchet *if its topology is given by a countable set of seminorms.*

We see that the space $\left(\mathbb{R}^\infty, \Pi_{i=1}^\infty \mathcal{T}_i\right)$ is a pre-Fréchet vector space. All pre-Fréchet spaces are metrizable. The distance in $\left(\mathbb{R}^\infty, \Pi_{i=1}^\infty \mathcal{T}_i\right)$ is defined as

$$\varrho(x, y) \stackrel{\text{def.}}{=} \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{[|x-y|]_i}{1+[|x-y|]_i} = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x^i - y^i|}{1+|x^i - y^i|}. \tag{2.4}$$

Metric (2.4) constitutes the same topology as the set of seminorms (2.3).

There is no norm establishing metric (2.4). This conclusion is the straightforward consequence of the fact that the maximal distance $\varrho(x, y)$ in metric (2.4) equals 1.

Definition 2.4. *[22] A complete pre-Fréchet space is called a* Fréchet space.

A theorem holds

Theorem 2.2. *[20] Let (V_i, ϱ_i), $i = 1, 2, \ldots$ are metric spaces and let $v = (v_1, v_2, \ldots)$, $u = (u_1, u_2, \ldots) \in \Pi_{i=1}^\infty V_i$ and*

$$\varrho(v, u) \stackrel{\text{def.}}{=} \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\varrho_i(v_i, u_i)}{1+\varrho_i(v_i, u_i)}. \tag{2.5}$$

The metric space $\left(\Pi_{i=1}^\infty V_i, \varrho\right)$ is complete iff all spaces (V_i, ϱ_i), $i = 1, 2, \ldots$ are complete.

The straightforward consequence of that theorem is the following.

Theorem 2.3. *The pre-Fréchet space $(\mathbb{R}^\infty, \Pi_{i=1}^\infty \mathcal{T}_i)$ is a Fréchet space.*

Let us recall the definition of a differentiable manifold [20].

Definition 2.5. *A differentiable n-dimensional manifold $\mathcal{M}$ of class C^r is a pair $(\mathcal{M}, \mathcal{A})$, where $\mathcal{M}$ is a Hausdorff space and $\mathcal{A} = \{(\mathcal{U}_z, \phi_z)\}_{z\in J}$ is a set of charts $(\mathcal{U}_z, \phi_z)$ and*

1. *$(\mathcal{U}_z, \phi_z)_{z\in J}$ is an open covering of $\mathcal{M}$ and $\phi_z : \mathcal{U}_z \to \mathcal{O}_z \subset \mathbb{R}^n$ are homeomorphisms,*
2. *mappings*

$$\phi_{zv} \stackrel{\text{def.}}{=} \phi_z \circ \phi_v^{-1} : \phi_v(\mathcal{U}_z \cap \mathcal{U}_v) \to \phi_z(\mathcal{U}_z \cap \mathcal{U}_v) \tag{2.6}$$

are r-times continuously differentiable and they are called transition functions.

By $\mathcal{O}_z$ we denote open subsets of $\mathbb{R}^n$.

Henceforth to shorten notation instead of writing $\left(\mathbb{R}^\infty, \Pi_{i=1}^\infty \mathcal{T}_i\right)$ we will put $\mathbb{R}^\infty$.

The main problem in a proof that the space $\mathbb{R}^\infty$ is some differentiable manifold is the fact that $\mathbb{R}^\infty$ is not a Banach space. To show that despite this obstacle $\mathbb{R}^\infty$ may be treated as a differentiable manifold we propose the following consideration.

1. Each element of the vector space $\mathbb{R}^\infty$ can be represented uniquely as an infinite series of real numbers $(x^1, x^2, \ldots)$. This suggests that there exist atlases on $\mathbb{R}^\infty$ containing only one chart $(\mathbb{R}^\infty, \phi)$ in which numbers x^i, $i = 1, 2, \ldots$ are coordinates of an arbitrary fixed point on $\mathbb{R}^\infty$.
2. From physical reasons which will be explained in the next section, it is sufficient to restrict our considerations to the atlas $\mathcal{A} = \{(\mathbb{R}^\infty, \phi_z)\}_{z\in J}$ on $\mathbb{R}^\infty$ such that each chart $(\mathbb{R}^\infty, \phi_z) \in \mathcal{A}$ covers the whole space $\mathbb{R}^\infty$ and moreover all mappings

$$\phi_{zv} = \phi_z \circ \phi_v^{-1} : \mathbb{R}^\infty \to \mathbb{R}^\infty$$

are linear bijections. The set of indices J can be finite, countable or uncountable. Each of bijections $\phi_{zv}(x^1, \ldots, x^i, \ldots) = (y^1, \ldots, y^i, \ldots)$ may be written in the following form:

$$\begin{array}{rcl} y^1 & = & a_{11}x^1 + a_{12}x^2 + \cdots \\ \vdots & & \qquad \ddots \\ y^i & = & a_{i1}x^1 + a_{i2}x^2 + \cdots \\ \vdots & & \qquad \ddots \end{array} \tag{2.7}$$

where $\forall_{i,j\in\mathcal{N}}\ a_{ij} \in \mathbb{R}$. To ensure convergence of sums standing at the right-hand side of the infinite system of equations (2.7) we require that for each index 'i' only the finite number of coefficients a_{ij} is different from 0. This condition holds for every mapping ϕ_{zv} so especially it is true also for the transformation $(y^1, \ldots, y^i, \ldots) \longrightarrow (x^1, \ldots, x^i, \ldots)$ inverse to (2.7).
3. Although the general definition of the derivative of the mapping ϕ_{zv} does not exist because of the lack of a norm in $\mathbb{R}^\infty$ we can precisely define partial derivatives

$$\frac{\partial y^i}{\partial x^j} \stackrel{\text{def.}}{=} \lim_{d\to 0} \frac{y^i(x^1, \ldots, x^j + d, \ldots) - y^i(x^1, \ldots, x^j, \ldots)}{d} \stackrel{(2.7)}{=} a_{ij}. \tag{2.8}$$

In this infinite-dimensional case we assume that the existence of all partial derivatives $\frac{\partial y^i}{\partial x^j}$ and $\frac{\partial x^j}{\partial y^i}$ (the proof for the inverse mapping is analogous) of an arbitrary range is sufficient to treat the mapping ϕ_{zv} as C^∞-differentiable.
4. A set of linear finite transformations of the form (2.7) constitutes a pseudogroup of transformations Γ (see [24]). The atlas $\mathcal{A} = \{(\mathbb{R}^\infty, \phi_z)\}_{z\in J}$ is compatible with the pseudogroup Γ. Since each atlas compatible with some subgroup is contained in a *unique* complete atlas of a manifold, starting from the atlas $\{(\mathbb{R}^\infty, \phi_z)\}_{z\in J}$ and the pseudogroup Γ we point out the complete atlas on $\mathbb{R}^\infty$.

From the construction presented in this section we conclude that $\mathbb{R}^\infty$ is really the differentiable manifold of a class C^∞.

3. The Weyl algebra

In this section we define and analyse some properties of a Weyl algebra. The method presented here is based on physical interpretation of this algebra. It is purely geometric in contrast to algebraic way proposed by Fedosov [6, 7]. We construct the Weyl algebra by a symmetric tensor product of spaces cotangent to some manifold.

Let $(\mathcal{M}, \omega)$ be a $2n-$ D symplectic manifold, $T^*_{\mathrm{p}}M$ is the cotangent space to $\mathcal{M}$ at a point p of $\mathcal{M}$ and $\mathcal{A} = \{(\mathcal{U}_z, \phi_z)\}_{z\in J}$ an atlas on $\mathcal{M}$.

Definition 3.1. *The space* $(T^*_{\mathrm{p}}M)^l, l \geqslant 1$ *is a symmetrized tensor product of* $\underbrace{T^*_{\mathrm{p}}M \odot \cdots \odot T^*_{\mathrm{p}}M}_{\text{l−times}}$. *It is spanned by*

$$v_{\mathbf{K_1}} \odot \cdots \odot v_{\mathbf{K_l}} \stackrel{\text{def.}}{=} \frac{1}{l!} \sum_{\text{all permutations}} v_{\sigma \mathbf{K_1}} \otimes \cdots \otimes v_{\sigma \mathbf{K_l}}, \tag{3.9}$$

where $v_{\mathbf{K_1}}, \ldots, v_{\mathbf{K_l}} \in T^*_{\mathrm{p}}M$. *For* $l = 0$ *we put* $(T^*_{\mathrm{p}}M)^0 \stackrel{\text{def.}}{=} \mathbb{R}$.

Each element $v \in (T^*_{\mathrm{p}}M)^l$ in a chart $(\mathcal{U}_z, \phi_z) \ni \mathrm{p}$ is uniquely represented by a sequence

$$v = \big(v_{1\ldots1}, \ldots, v_{i_1\ldots i_l}, \ldots, v_{2n\ldots2n}\big). \tag{3.10}$$

For indices the relation holds $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{l-1} \leqslant i_l$. The number of elements of the sequence (3.10) is equal to $\frac{(2n+l-1)!}{(2n-1)!\,l!}$. This is the straightforward consequence of the fact that

$$\underbrace{\sum_{k_1=1}^{m} \sum_{k_2=k_1}^{m} \cdots \sum_{k_l=k_{l-1}}^{m}}_{\text{l−times}} 1 = \frac{(m+l-1)!}{l!(m-1)!}. \tag{3.11}$$

Introducing the distance $\varrho_l(v, u)$ between two elements of $(T^*_{\mathrm{p}}M)^l$ as

$$\varrho_l(v,u) \stackrel{\text{def.}}{=} \begin{cases} |v-u| & \text{for} \quad l = 0, \\ \sum_{\text{all } i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{l-1} \leqslant i_l} |v_{i_1 i_2 \cdots i_{l-1} i_l} - u_{i_1 i_2 \cdots i_{l-1} i_l}| & \text{for} \quad l > 0, \end{cases} \tag{3.12}$$

we define a metric structure in $(T^*_{\mathrm{p}}M)^l$. The distance between v and u depends on the choice of the system of coordinates on the manifold $\mathcal{M}$.

A set of balls

$$K_l(v,r) = \{u \in (T^*_{\mathrm{p}}M)^l, \varrho_l(v,u) < r\}, \qquad v \in (T^*_{\mathrm{p}}M)^l, \quad r > 0 \tag{3.13}$$

introduces a topology $\mathcal{T}_l$ on $(T^*_{\mathrm{p}}M)^l$ so $((T^*_{\mathrm{p}}M)^l, \mathcal{T}_l)$ is a topological space. Although the distance $\varrho_l(v, u)$ depends on the choice of a chart on the manifold $\mathcal{M}$, the topology $\mathcal{T}_l$ is independent of it. This conclusion becomes obvious if we note that the topology defined by open balls (3.13) is the same as the topology established by cubes

$$|v_{1\ldots1} - u_{1\ldots1}| \times \cdots \times |v_{2n\ldots2n} - u_{2n\ldots2n}|.$$

Moreover $((T^*_{\mathrm{p}}M)^l, \mathcal{T}_l)$ is a complete vector space. It is also a differentiable manifold modelled on a Banach space $(\mathbb{R}^{\frac{(2n+l-1)!}{(2n-1)!l!}}, \|\cdot\|)$ with a norm

$$\|x\| \stackrel{\text{def.}}{=} \sum_{i=1}^{\frac{(2n+l-1)!}{(2n-1)!\,l!}} |x^i|.$$

Definition 3.2. *A* preWeyl vector space $P^*_{\mathrm{p}}M$ *at a point* $\mathrm{p} \in \mathcal{M}$ *is the direct sum*

$$P^*_{\mathrm{p}}M \stackrel{\text{def.}}{=} \bigoplus_{l=0}^{\infty} ((T^*_{\mathrm{p}}M)^l \oplus (T^*_{\mathrm{p}}M)^l).$$

Beside $\bigoplus_{l=0}^{\infty}$ also another direct sum appears because components of tensors which are used in physics are in general complex numbers. The preWeyl vector space is a topological space with the Tichonov topology. Construction of a metric and topology is analogous to that presented in the previous section.

$$\varrho(v,u) \stackrel{\text{def.}}{=} \sum_{l=0}^{\infty} \left(\frac{1}{2^{l+2}} \frac{\varrho_l(\mathrm{Re}(v),\mathrm{Re}(u))}{1+\varrho_l(\mathrm{Re}(v),\mathrm{Re}(u))} + \frac{1}{2^{l+2}} \frac{\varrho_l(\mathrm{Im}(v),\mathrm{Im}(u))}{1+\varrho_l(\mathrm{Im}(v),\mathrm{Im}(u))} \right) \tag{3.14}$$

for each $v, u \in P^*_{\mathrm{p}}M$. The distances $\varrho_l(\mathrm{Re}(v),\mathrm{Re}(u))$ and $\varrho_l(\mathrm{Im}(v),\mathrm{Im}(u))$ are computed between real and imaginary parts of components of v, u belonging to $(T^*_{\mathrm{p}}M)^l \oplus (T^*_{\mathrm{p}}M)^l$. Again, although metric (3.14) depends on the choice of a chart on $\mathcal{M}$, the Tichonov topology on $P^*_{\mathrm{p}}M$ is independent of it.

Due to theorem 2.2 the preWeyl space is also a Fréchet space.

Definition 3.3. *[14] Let λ be a fixed real number and V some vector space. A* formal series *in the formal parameter λ is every expression of the form*

$$v[[\lambda]] = \sum_{\mathbf{K}=0}^{\infty} \lambda^{\mathbf{K}} v_{\mathbf{K}}, \qquad \text{where} \quad \forall_{\mathbf{K}} v_{\mathbf{K}} \in V. \tag{3.15}$$

The set of formal series $v[[\lambda]]$ constitutes a vector space.

Addition means vector summation of elements of the same power of λ and multiplication by a scalar $a \in \mathbb{C}$ is a multiplication of each vector standing on the right-hand side of (3.15) by a, i.e.

$$u[[\lambda]] + v[[\lambda]] = \sum_{\mathbf{K}=0}^{\infty} \lambda^{\mathbf{K}} (u_{\mathbf{K}} + v_{\mathbf{K}}) \tag{3.16}$$

and

$$a \cdot v[[\lambda]] = \sum_{\mathbf{K}=0}^{\infty} \lambda^{\mathbf{K}} (a v_{\mathbf{K}}). \tag{3.17}$$

A vector space of formal series over the vector space V in the formal parameter λ we will denote by $V[[\lambda]]$. The space $V[[\lambda]]$ may be treated as a direct sum

$$V[[\lambda]] = \bigoplus_{i=0}^{\infty} V_i, \qquad V_i = V \quad \text{for every } i. \tag{3.18}$$

We introduce the formal series over the preWeyl vector space as follows.

Definition 3.4. *A* Weyl vector space $P^*_{\mathrm{p}}M[[\hbar]]$ *is the vector space over the preWeyl vector space $P^*_{\mathrm{p}}M$ in the formal parameter $\hbar$.*

For physical applications we usually identify the parameter $\hbar$ with the Planck constant. Terms standing at the kth power of $\hbar^k$ and belonging to the direct sum $(T^*_{\mathrm{p}}M)^l \oplus (T^*_{\mathrm{p}}M)^l$ we will denote by $v[k,l]$. Now each element of $P^*_{\mathrm{p}}M[[\hbar]]$ may be written in the form

$$v = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hbar^k v[k,l]. \tag{3.19}$$

For $l = 0$ we put $\sum_{k=0}^{\infty} \hbar^k v[k,0]$, $v[k,0] \in \mathbb{C}$. Again using the Tichonov procedure we equip the Weyl space with a topological structure. The Weyl space is a Fréchet space (see theorem 2.2) with the metric

$$\varrho(u,v) \stackrel{\text{def.}}{=} \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \frac{\varrho_k(u,v)}{1+\varrho_k(u,v)}, \tag{3.20}$$

where by $\varrho_k(u,v)$ we understand the distance (3.14) computed between parts of u and v standing at $\hbar^k$.

Definition 3.5. *[7] The* degree $\deg(v[k,l])$ *of the component* $v[k,l]$ *of the Weyl vector space* $P^*_{\mathrm{p}}M[[\hbar]]$ *is the sum* $2k+l$.

At the beginning of this paragraph we propose the following convention—coordinates of a vector $v[k,l]$ we will denote by $v[k,l]_{i_1\ldots i_l}$. To show that $P^*_{\mathrm{p}}M[[\hbar]]$ is a differentiable manifold first we introduce one chart $(P^*_{\mathrm{p}}M[[\hbar]],\phi)$ covering the whole space. The mapping ϕ is defined as follows:

$$\phi(v) = (\mathrm{Re}(v[0,0]), \mathrm{Im}(v[0,0]), \mathrm{Re}(v[0,1]_1), \mathrm{Im}(v[0,1]_1), \ldots). \qquad (3.21)$$

Let $(\phi(v))_d$ denote the dth element of sequence (3.21). Elements in this sequence are ordered according to the following rules:

1. $\deg(\phi(v))_{d_1} > \deg(\phi(v))_{d_2} \Longrightarrow d_1 > d_2$.
2. For the same degree if the power of $\hbar$ in $(\phi(v))_{d_1}$ is higher than in $(\phi(v))_{d_2}$ then $d_1 > d_2$.
3. For the same degree and power of $\hbar$ we order elements as in (3.10) taking into account the indices of a tensor.
4. For two terms of the same degree, the same power of $\hbar$ and the same tensor indices the real part precedes the imaginary one.

A formula connecting the position of $v[k,l]_{i_1\ldots i_l}$ in sequence (3.21) with indices $k, i_1\ldots i_l$ is rather complicated. The reader may find it in the appendix.

The chart $(P^*_{\mathrm{p}}M[[\hbar]],\phi)$ is determined by the choice of a chart on the symplectic manifold $\mathcal{M}$ because numbers $(\phi(v))_d$ are components of tensors in a natural basis given by coordinates on $\mathcal{M}$. Let us cover the Weyl space with an atlas $\mathcal{A} = \{(\mathcal{U}_z,\phi_z)\}_{z\in J}$ consisting of all natural charts. Each mapping ϕ_z satisfies the ordering rule (3.21) and covers the whole Weyl space.

1. $\forall_{z\in J}\phi_z$ are homeomorphisms $(P^*_{\mathrm{p}}M[[\hbar]],\phi)\to\mathbb{R}^\infty$.
2. Mappings $\phi_{zv} = \phi_z\circ\phi_v^{-1} : \mathbb{R}^\infty\to\mathbb{R}^\infty$ are linear bijections of the kind (2.7). Moreover, each arbitrary fixed element $(\phi_z(v))_i$ depends linearly only on terms $(\phi_v(v))_j$ characterized by the same power of $\hbar$, the same tensor range and belonging to the same real or imaginary part of v. Thus we conclude that all partial derivatives $\frac{\partial(\phi_z(v))_j}{\partial(\phi_v(v))_i}$ exist. Moreover, for a fixed j only a finite number of those partial derivatives do not vanish.

Taking into account facts presented above we say that the Weyl space $P^*_{\mathrm{p}}M[[\hbar]]$ is a differentiable manifold.

For physical reasons (see [7]) the Weyl space may be equipped with a structure of an algebra. Let $X_{\mathrm{p}}\in T_{\mathrm{p}}M$ be some fixed vector from the space $T_{\mathrm{p}}M$ tangent to $\mathcal{M}$ at a point p. Components of X_{p} in the natural basis $\left\{\frac{\partial}{\partial q^1},\ldots,\frac{\partial}{\partial q^{2n}}\right\}$ we denote byX^i_{p}. It is clear that for every $v[k,l]\in P^*_{\mathrm{p}}M[[\hbar]]$ the acting

$$v[k,l](\underbrace{X_{\mathrm{p}},\ldots,X_{\mathrm{p}}}_{\text{l−times}}) = v[k,l]_{i_1\ldots i_l}X^{i_1}_{\mathrm{p}}\cdots X^{i_l}_{\mathrm{p}}$$

is a complex number and we can treat $v[k,l](\underbrace{X_{\mathrm{p}},\ldots,X_{\mathrm{p}}}_{\text{l−times}})$ as a polynomial of the lth degree in components of the vector X_{p}.

Thus elements of the Weyl space $P^*_{\mathrm{p}}M[[\hbar]]$ are mappings

$$\begin{aligned} &v(X_{\mathrm{p}}) : \mathbb{R}^{2n}\to\mathbb{C}[[\hbar]],\\ &v(X_{\mathrm{p}}) \stackrel{\text{def.}}{=} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\hbar^k v[k,l]_{i_1\ldots i_l}X^{i_1}_{\mathrm{p}}\cdots X^{i_l}_{\mathrm{p}}.\end{aligned} \qquad (3.22)$$

A symbol $\mathbb{C}[[\hbar]]$ denotes a space of formal series over $\mathbb{C}$.

Let us go to the definition of ∘-product in the Weyl space. By a derivative $\frac{\partial v}{\partial X^i_{\mathrm{p}}}$ we understand the formal sum (ordered by powers of $\hbar^k$) of partial derivatives of polynomials $v[k,l]_{i_1\ldots i_l} X^{i_1}_{\mathrm{p}} \cdots X^{i_l}_{\mathrm{p}}$. Derivation $\frac{\partial}{\partial X^i_{\mathrm{p}}}$ does not influence powers of $\hbar$. The definition of derivation presented here is formal because in the space $\mathbb{C}[[\hbar]]$ a norm is not defined.

Definition 3.6. *[7] The product* $\circ : P^*_{\mathrm{p}}M[[\hbar]] \times P^*_{\mathrm{p}}M[[\hbar]] \to P^*_{\mathrm{p}}M[[\hbar]]$ *of two elements* $v, u \in P^*_{\mathrm{p}}M[[\hbar]]$ *is such an element* $w \in P^*_{\mathrm{p}}M[[\hbar]]$ *that for each* $X_{\mathrm{p}} \in T_{\mathrm{p}}M$ *the equality holds*

$$w(X_{\mathrm{p}}) = v(X_{\mathrm{p}}) \circ u(X_{\mathrm{p}})$$

$$\stackrel{\text{def.}}{=} \sum_{t=0}^{\infty} \frac{1}{t!}\left(\frac{\mathrm{i}\hbar}{2}\right)^t \omega^{i_1 j_1}\cdots\omega^{i_t j_t} \frac{\partial^t v(X_{\mathrm{p}})}{\partial X^{i_1}_{\mathrm{p}}\ldots\partial X^{i_t}_{\mathrm{p}}} \frac{\partial^t u(X_{\mathrm{p}})}{\partial X^{j_1}_{\mathrm{p}}\ldots\partial X^{j_t}_{\mathrm{p}}}. \tag{3.23}$$

The pair $(P^*_{\mathrm{p}}M[[\hbar]], \circ)$ is a noncommutative associative algebra called the *Weyl algebra*. By ω^{ij} we understand components of the tensor inverse to the symplectic form in a point p, i.e. the relation holds

$$\omega^{ij}\omega_{jk} = \delta^i_k.$$

Here we mention some properties of the ∘-product in $(P^*_{\mathrm{p}}M[[\hbar]], \circ)$. More information can be found in [7, 8, 25]. It is worth emphasizing that the first of the presented properties becomes clear thanks to our geometrical approach to the Weyl algebra.

1. The ∘-product is independent of the chart.
2. The ∘-multiplication is associative but nonAbelian.
3. $\forall_{v,u\in(P^*_{\mathrm{p}}M[[\hbar]],\circ)}$ the relation holds $\deg(v \circ u) = \deg(v) + \deg(u)$.

Let us show the continuity of the ∘-product in the Tichonov topology in $(P^*_{\mathrm{p}}M[[\hbar]], \circ)$. Analogous to formula (2.3) in an arbitrary fixed chart $(P^*_{\mathrm{p}}M[[\hbar]], \phi)$ we introduce seminorms as

$$\forall_{v\in(P^*_{\mathrm{p}}M[[\hbar]],\circ)}\forall_{i\in\mathcal{N}} [|v|]_i \stackrel{\text{def.}}{=} |(\phi(v))_i|. \tag{3.24}$$

Metric (3.20) is now expressed by seminorms $\{[|\cdot|]_i\}_{i\in\mathcal{N}}$ as (compare with (2.4))

$$\varrho(v,u) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{[|v-u|]_i}{1+[|v-u|]_i}. \tag{3.25}$$

Let us consider two sequences $\{v_{\mathbf{K}}\}_{\mathbf{K}\in\mathcal{N}}, \{u_{\mathbf{J}}\}_{\mathbf{J}\in\mathcal{N}}$ elements $v_{\mathbf{K}}, u_{\mathbf{J}} \in (P^*_{\mathrm{p}}M[[\hbar]], \circ)$ such that

$$\lim_{\mathbf{K}\to\infty} v_{\mathbf{K}} = \lim_{\mathbf{J}\to\infty} u_{\mathbf{J}} = \Theta. \tag{3.26}$$

This relation implies that for every $i \in \mathcal{N}$

$$\lim_{\mathbf{K}\to\infty} [|v_{\mathbf{K}}|]_i = \lim_{\mathbf{J}\to\infty} [|u_{\mathbf{J}}|]_i = 0. \tag{3.27}$$

The ∘-product is continuous if for every two sequences $\{v_{\mathbf{K}}\}_{\mathbf{K}\in\mathcal{N}}, \{u_{\mathbf{J}}\}_{\mathbf{J}\in\mathcal{N}}$ fulfilling the condition (3.26) and for every two $a, b \in (P^*_{\mathrm{p}}M[[\hbar]], \circ)$ the equality holds

$$\lim_{\mathbf{J},\mathbf{K}\to\infty} ((a + v_{\mathbf{K}}) \circ (b + u_{\mathbf{J}})) = a \circ b. \tag{3.28}$$

Formula (3.28) is equivalent to the system of equations

$$\lim_{\mathbf{K}\to\infty} v_{\mathbf{K}} \circ b = \Theta, \tag{3.29}$$

$$\lim_{\mathbf{J}\to\infty} a \circ u_{\mathbf{J}} = \Theta, \tag{3.30}$$

$$\lim_{\mathbf{J},\mathbf{K}\to\infty} v_{\mathbf{K}} \circ u_{\mathbf{J}} = \Theta. \tag{3.31}$$

Let us consider an element $(v_{\mathbf{K}} \circ b)[r, l]_{i_1\ldots i_l}$. Its degree equals $2r + l$. From the definition of the $\circ$-product (3.23) and its third property (look at previous page) we see that $(v_{\mathbf{K}} \circ b)[r, l]_{i_1\ldots i_l}$ is a finite linear combination of components of vectors $v_{\mathbf{K}}$ such that $\deg v_{\mathbf{K}} \leqslant 2r + l$. Since that for $\mathbf{J} \to \infty$ equality (3.29) holds. Using the same method we prove relations (3.30) and (3.31).

4. The Weyl bundle

Until now we have worked with the Weyl algebra $(P^*M_{\mathrm{p}}[[\hbar]], \circ)$ at an arbitrary fixed point p belonging to the symplectic manifold $\mathcal{M}$. Now we are going to analyse a collection of Weyl algebras taken for all points of $\mathcal{M}$.

Definition 4.1. *A* Weyl bundle *is a triplet* $(\mathcal{P}^*\mathcal{M}[[\hbar]], \pi, \mathcal{M})$*, where*

$$\mathcal{P}^*\mathcal{M}[[\hbar]] \stackrel{\text{def.}}{=} \bigcup_{\mathrm{p}\in\mathcal{M}} P^*M_{\mathrm{p}}[[\hbar]] \tag{4.32}$$

is a differentiable manifold called a total space, $\mathcal{M}$ *is a base space and* $\pi : \mathcal{P}^*\mathcal{M}[[\hbar]] \to \mathcal{M}$ *a projection.*

Elements of the Weyl bundle $\mathcal{P}^*\mathcal{M}[[\hbar]]$ can be thought of as the pairs (p, v), where $v \in (P^*M_{\mathrm{p}}[[\hbar]], \circ)$. The projection π assigns a point p to the pair (p, v).

The Weyl algebra $(P^*_{\mathrm{p}}M[[\hbar]], \circ)$ considered in the previous section is related to the point $\mathrm{p} \in \mathcal{M}$. To define a fibre in the Weyl bundle $\mathcal{P}^*\mathcal{M}[[\hbar]]$ it is required to introduce the $\circ$-product in the topological vector space $\mathbb{R}^\infty$. However, the explicit form of a product $u \circ v$, where $u, v \in \mathbb{R}^\infty$ are of form (3.21), is complicated and useless for practical purposes. But in fact it is sufficient to be aware that the algebra $(\mathbb{R}^\infty, \circ)$ exists.

Definition 4.1 contains a statement that the Weyl bundle $\mathcal{P}^*\mathcal{M}[[\hbar]]$ is a C^∞ differentiable manifold. To prove this fact we define a topology on $\mathcal{P}^*\mathcal{M}[[\hbar]]$ first.

Let $\mathcal{A} = \{(\mathcal{U}_z, \phi_z)\}_{z\in J}$ be an atlas on the $\mathcal{M}$. By the definition

$$P' \stackrel{\text{def.}}{=} \bigcup_{z\in J} (\mathcal{U}_z \times \mathbb{R}^\infty \times \{z\}) \subset \mathcal{M} \times \mathbb{R}^\infty \times J.$$

The set J is equipped with the discrete topology. The topology in $\mathcal{M} \times \mathbb{R}^\infty \times J$ is constructed according to definition 2.1 so the Cartesian product $\mathcal{M} \times \mathbb{R}^\infty \times J$ has a Tichonov topology. Since $\bigcup_{z\in J} (\mathcal{U}_z \times \mathbb{R}^\infty \times \{z\})$ is a subset of $\mathcal{M} \times \mathbb{R}^\infty \times J$, we equip it with the induced topology.

Now in P' we establish the equivalence relation

$$(\mathrm{p}, v, z) \sim (\mathrm{p}', v', d) \quad \text{iff} \tag{4.33}$$

1. $\mathrm{p} = \mathrm{p}'$,
2. v' is the image of v in the mapping induced by the change of coordinates $(\mathcal{U}_z, \phi_z) \to (\mathcal{U}_d, \phi_d)$ on $\mathcal{M}$. It is important to remember that the tensor ω^{ij} also transforms under that change of coordinates.

$\mathcal{P}^*\mathcal{M}[[\hbar]] \stackrel{\text{def.}}{=} P'/\sim$ possesses the quotient topology so the Weyl bundle is a topological space.

Let us consider two projections:

1. a canonical projection $G : P' \to P'/\sim$ defined as

$$\forall_{(\mathrm{p},v,z)\in P'} G((\mathrm{p}, v, z)) = [(\mathrm{p}, v, z)]. \tag{4.34}$$

Since the topology on $\mathcal{P}^*\mathcal{M}[[\hbar]]$ is the quotient one, the canonical projection G and its inverse G^{-1} are continuous mappings [21]. A symbol $[\cdot]$ denotes the equivalence class;

2. a projection $L : P' \to \mathcal{M}$ fulfilling the equation

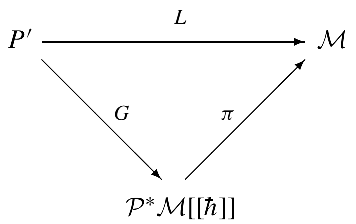
$$\forall_{(\mathrm{p},v,z)\in P'} L((\mathrm{p}, v, z)) = \mathrm{p}. \tag{4.35}$$

In the topology of a Cartesian product on $\mathcal{M} \times \mathbb{R}^\infty \times J$ a projection $\mathrm{Pr} : \mathcal{M} \times \mathbb{R}^\infty \times J \to \mathcal{M}$ defined as

$$\forall_{(\mathrm{p},v,z)\in \mathcal{M}\times\mathbb{R}^\infty\times J} \mathrm{Pr}((\mathrm{p}, v, z)) = \mathrm{p}$$

is continuous from the definition of the Tichonov topology. The projection L is nothing but $\mathrm{Pr}|_{P'}$ and the topology on P' has been induced from $\mathcal{M} \times \mathbb{R}^\infty \times J$ so the mapping L is continuous.

Projections G and L preserve the point p so we can draw a commutative diagram



The mapping $\pi \stackrel{\text{def.}}{=} L \circ G^{-1}$ as a product of two continuous mappings is also continuous. The equality holds

$$G(\mathcal{U}_z \times \mathbb{R}^\infty \times \{z\}) = \pi^{-1}(\mathcal{U}_z). \tag{4.36}$$

Relation (4.36) constitutes a bijection between $\mathcal{U}_z \times \mathbb{R}^\infty \times \{z\}$ and $\pi^{-1}(\mathcal{U}_z)$. Moreover G and G^{-1} are continuous mappings so we conclude that

$$G|_{\mathcal{U}_z\times\mathbb{R}^\infty\times\{z\}} : \mathcal{U}_z \times \mathbb{R}^\infty \times \{z\} \to \pi^{-1}(\mathcal{U}_z) \tag{4.37}$$

is a homeomorphism.

The next step is to prove that $\mathcal{P}^*\mathcal{M}[[\hbar]]$ is a Hausdorff space. Let $\mathrm{q}_1, \mathrm{q}_2 \in \mathcal{P}^*\mathcal{M}[[\hbar]]$ and $\pi(\mathrm{q}_1) \neq \pi(\mathrm{q}_2)$. The manifold $\mathcal{M}$ is a Hausdorff space so we can always choose two neighbourhoods $\mathcal{V}_1, \mathcal{V}_2$ such that $\pi(\mathrm{q}_1) \in \mathcal{V}_1, \pi(\mathrm{q}_2) \in \mathcal{V}_2$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$.

Let us introduce the identification $\mathrm{Id} : \mathcal{U}_z \times \mathbb{R}^\infty \times \{z\} \cong \mathcal{U}_z \times \mathbb{R}^\infty$. Superposition of two mappings

$$\mathrm{Id} \circ \left(G|_{\mathcal{U}_z\times\mathbb{R}^\infty\times\{z\}}\right)^{-1} : \pi^{-1}(\mathcal{U}_z) \to \mathcal{U}_z \times \mathbb{R}^\infty \tag{4.38}$$

is a homeomorphism so $\pi^{-1}(\mathcal{V}_1)$ and $\pi^{-1}(\mathcal{V}_2)$ are neighbourhoods of $\mathrm{q}_1, \mathrm{q}_2$ respectively and, moreover, $\pi^{-1}(\mathcal{V}_1) \cap \pi^{-1}(\mathcal{V}_2) = \emptyset$.

In the case $\mathrm{q}_1, \mathrm{q}_2 \in \mathcal{P}^*\mathcal{M}[[\hbar]]$, $\mathrm{q}_1 \neq \mathrm{q}_2$ and $\pi(\mathrm{q}_1) = \pi(\mathrm{q}_2)$ we have $\mathrm{q}_1, \mathrm{q}_2 \in \pi^{-1}(\mathcal{U}_z)$ for some $z \in J$. But from the fact that (4.38) is a homeomorphism we conclude that $\pi^{-1}(\mathcal{U}_z)$ is a Hausdorff space so there exist separate neighbourhoods of q_1 and q_2.

So it has been proved that the Weyl bundle is a Hausdorff space.

The Cartesian product of the open subset $\mathcal{U}_z$ and the fibre $\mathbb{R}^\infty$ is homeomorphic to $\mathcal{O}_z \times \mathbb{R}^\infty$, where $\mathcal{O}_z$ is an open subset of $\mathbb{R}^{2n}$. We denote that homeomorphism by $\varsigma_z : \mathcal{U}_z \times \mathbb{R}^\infty \to \mathcal{O}_z \times \mathbb{R}^\infty$. The mapping ς_z is determined by a chart $(\mathcal{U}_z, \phi_z)$. Namely

$$\varsigma_z(\mathrm{p}, v) = \left(\phi_z(\mathrm{p}), T_{\phi_z}(v)\right). \tag{4.39}$$

The symbol T_{ϕ_z} denotes taking the natural components of v. Remember that also the tensor ω^{ij} from definition (3.23) transforms under ς_z. The homeomorphism

$$\begin{aligned} &\varphi_z : \pi^{-1}(\mathcal{U}_z) \to \mathcal{O}_z \times \mathbb{R}^\infty, \\ &\varphi_z \stackrel{\text{def.}}{=} \varsigma_z \circ \text{Id} \circ G^{-1} \end{aligned} \tag{4.40}$$

establishes a differential structure on the Weyl bundle $\mathcal{P}^*\mathcal{M}[[\hbar]]$. Indeed an atlas on $\mathcal{P}^*\mathcal{M}[[\hbar]]$ is the set of charts $\{(\pi^{-1}(\mathcal{U}_z), \varphi_z)\}_{z\in J}$. Mappings $\varphi_z \circ \varphi_v^{-1}$ are transition functions and they are C^∞-differentiable in a sense that all of their partial derivatives exist.

Thus we conclude that $\mathcal{P}^*\mathcal{M}[[\hbar]]$ is a C^∞-class infinite-dimensional differentiable manifold.

The Weyl bundle is an example of a vector bundle. Apart from elements described above the definition of a vector bundle (for details see [23]) consists of a structure group $\mathcal{G}$ being a Lie group and local trivializations.

If by $GL(2n, \mathbb{R})$ we denote the group of real automorphisms of the cotangent space $T^*_{\mathrm{p}}M$, the structure group of the fibre is

$$\mathcal{G} \stackrel{\text{def.}}{=} \bigoplus_{z=0}^{\infty} \bigoplus_{k=0}^{[\frac{z}{2}]} \left(\underbrace{GL(2n, \mathbb{R}) \otimes \cdots \otimes GL(2n, \mathbb{R})}_{(z-2k)\text{-times}} \oplus \underbrace{GL(2n, \mathbb{R}) \otimes \cdots \otimes GL(2n, \mathbb{R})}_{(z-2k)\text{-times}} \right). \tag{4.41}$$

Moreover, when the element $v_{i_1 i_2}$ transforms under the element $g \in GL(2n, \mathbb{R}) \otimes GL(2n, \mathbb{R})$ then ω^{ij} transforms under $g^{-1} \in GL(2n, \mathbb{R}) \otimes GL(2n, \mathbb{R})$.

Mappings $\text{Id} \circ (G|_{\mathcal{U}_z \times \mathbb{R}^\infty \times \{z\}})^{-1}$ (see (4.37)) are local trivializations of the Weyl bundle because they map $\pi^{-1}(\mathcal{U}_z)$ onto the direct product $\mathcal{U}_z \times \mathbb{R}^\infty$.

Since the fibre $\mathbb{R}^\infty$ is not only a vector space but also an algebra, the Weyl bundle is an example of the algebra bundle.

Knowing that the Weyl bundle is a differentiable manifold we can easily define smooth sections of it or introduce a parallel transport on $\mathcal{P}^*\mathcal{M}[[\hbar]]$. Physical application of these quantities will be explained in the last part of our contribution.

5. Connections in the Weyl bundle

Now we are ready to present the construction of a connection in the Weyl bundle. This construction plays crucial role in physical applications of mathematics contained in our contribution. As before we begin with some general definitions and later apply them to the Weyl algebra bundle.

Definition 5.2. *[27] Suppose $(E, \pi, \mathcal{M})$ is a vector bundle over a manifold $\mathcal{M}$ and $C^\infty(E)$ is a set of smooth sections of E over $\mathcal{M}$. An* exterior covariant derivative *on the bundle $(E, \pi, \mathcal{M})$ is a map*

$$\tilde{\partial} : C^\infty(E) \longrightarrow C^\infty(\mathcal{T}^*\mathcal{M} \otimes E),$$

which satisfies the following conditions:

1. for any $u, v \in C^\infty(E)$

$$\tilde{\partial}(u + v) = \tilde{\partial}(u) + \tilde{\partial}(v), \tag{5.42}$$

2. for any $v \in C^\infty(E)$ and any $f \in C^\infty(\mathcal{M})$

$$\tilde{\partial}(fv) = \mathrm{d}f \otimes v + f \cdot \tilde{\partial}v. \tag{5.43}$$

Remarks

1. The map $\tilde{\partial}$ is also called a connection (see [6, 7, 27, 28]). We keep the convention used in our earlier papers and by 'connection' understand only a form determining exterior covariant derivative of a local frame field (see formula (5.51)).
2. By $\mathcal{T}^*\mathcal{M}$ the cotangent bundle over the manifold $\mathcal{M}$ is denoted.

Let us introduce a new symbol. $\Lambda^p(\mathcal{M})$ is a bundle of p-forms over $\mathcal{M}$. Smooth sections of the tensor product $E \otimes \Lambda^p(\mathcal{M})$ are known as p-forms on $\mathcal{M}$ of values in E or E-valued p-forms. Since $C^\infty(E \otimes \Lambda^p(\mathcal{M}))$ is a module over $C^\infty(\mathcal{M})$, there is a module isomorphism ([28])

$$C^\infty(E) \otimes C^\infty(\Lambda^p(\mathcal{M})) \longrightarrow C^\infty(E \otimes \Lambda^p(\mathcal{M})) \tag{5.44}$$

which we shall denote by $v \otimes f \to f \cdot v$ or simply fv. It means the second condition from definition 5.2 takes the form

$$\tilde{\partial}(fv) = \mathrm{d}f \cdot v + f \cdot \tilde{\partial}v. \tag{5.45}$$

We are ready to find a more 'operational' form of the map $\tilde{\partial}$. Let a system of vectors $e_1, e_2, \ldots, e_g$ (in general an infinite one) constitute a basis of the fibre E_{p} at a point $\mathrm{p} \in \mathcal{M}$. Suppose that a matrix of base vectors is given by

$$e = [e_1 e_2 \ldots e_g]. \tag{5.46}$$

Moreover let

$$A = \begin{bmatrix} a_1^1 & a_2^1 & \ldots & a_g^1 \\ a_1^2 & a_2^2 & \ldots & a_g^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^g & a_2^g & \ldots & a_g^g \end{bmatrix} \tag{5.47}$$

be a nonsingular real matrix. In the case when $g = \infty$ it is supposed that in each row only a finite number of terms is different from 0. The same assumption is true for the inverse matrix A^{-1}. Each system of versors e' such that

$$e' = e \cdot A \tag{5.48}$$

is also a basis of E_{p}. A vector $v \in E_{\mathrm{p}}$ in the basis e is represented by the 1-column matrix

$$v = \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^g \end{bmatrix}. \tag{5.49}$$

Under the linear map determined by the matrix A the transformation rule for v is given by the formula

$$v' = A^{-1} \cdot v. \tag{5.50}$$

Having given the basis e of the fibre at the point p we need to propagate it smoothly on the whole neighbourhood $\mathrm{p} \in \mathcal{U} \subset \mathcal{M}$ of the point p. To do it we choose smooth sections $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_g}$ of the bundle E over $\mathcal{U}$ in such a manner that at each point $\mathrm{q} \in \mathcal{U}\ \forall_i\ \mathbf{e_i}|_{\mathrm{q}} = e_i$. That set of sections constitutes a *local frame field* of E on $\mathcal{U}$. At every point $\mathrm{q} \in \mathcal{U}$ the system of vectors $\mathrm{d}q^j \otimes e_i$, $1 \leqslant j \leqslant 2n$, $1 \leqslant i \leqslant g$ forms a basis of $T^*_{\mathrm{p}}M \otimes E$. Since an exterior covariant derivative $D\mathbf{e_i}$ is a local section of $\mathcal{T}^*\mathcal{M} \otimes E$, we can write

$$\tilde{\partial}\mathbf{e_i} = \sum_{j=1}^{2n} \sum_{k=1}^{g} \Gamma_{ij}^k \, \mathrm{d}q^j \otimes \mathbf{e_k}. \tag{5.51}$$

Coefficients Γ_{ij}^k are smooth real functions on $\mathcal{U}$ and they are called *connection coefficients* on E over $\mathcal{U}$. It is always possible to propagate a connection on the whole bundle (see [27]).

Introducing the connection matrix

$$\varpi_i^k \stackrel{\text{def.}}{=} \sum_{j=1}^{2n} \Gamma_{ij}^k \, \mathrm{d}q^j, \tag{5.52}$$

$$\varpi = \begin{bmatrix} \varpi_1^1 & \varpi_2^1 & \cdots \\ \varpi_1^2 & \varpi_2^2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \tag{5.53}$$

we see that

$$\tilde{\partial}\mathbf{e_i} = \sum_{k=1}^{g} \varpi_i^k \otimes \mathbf{e_k} \stackrel{(5.44)}{=} \sum_{k=1}^{g} \varpi_i^k \mathbf{e_k}$$

or, equivalently, in terms of matrices

$$\tilde{\partial}\mathbf{e} = \varpi \cdot \mathbf{e}.$$

It means that the exterior covariant derivative of a section $v \in C^\infty(E)$ over $\mathcal{U}$ equals

$$\tilde{\partial} v = D\left(\sum_{i=1}^{g} v^i \mathbf{e_i}\right) \stackrel{(5.42),(5.43)}{=} \sum_{i=1}^{g} \mathrm{d}v^i \otimes \mathbf{e_i} + \sum_{i,k=1}^{g} v^i \varpi_i^k \otimes \mathbf{e_k}$$

$$\stackrel{(5.44)}{=} \sum_{i=1}^{g} \mathrm{d}v^i \mathbf{e_i} + \sum_{i,k=1}^{g} v^i \varpi_i^k \mathbf{e_k}. \tag{5.54}$$

As can be proved [28], the connection matrix ϖ transforms according to the rule

$$\varpi' = A^{-1}\,\mathrm{d}A + A^{-1}\varpi A. \tag{5.55}$$

Till now we were able to differentiate only vectors from the bundle E. But it is possible to extend the operation $\tilde{\partial}$ to p-forms with values in E.

Theorem 5.1 [28]. *There is a unique operator*

$$\partial : C^\infty(E \otimes \Lambda^p(\mathcal{M})) \longrightarrow C^\infty(E \otimes \Lambda^{p+1}(\mathcal{M}))$$

satisfying

1. *for all* $f \in C^\infty(\Lambda^q(\mathcal{M}))$, $v \in C^\infty(E \otimes \Lambda^p(\mathcal{M}))$

$$\partial(f \wedge v) = \mathrm{d}f \wedge v + (-1)^q f \wedge \partial v, \tag{5.56}$$

2. *for* $v \in C^\infty(E)$

$$\tilde{\partial} v = \partial v. \tag{5.57}$$

It is easy to check that for $v^i \in C^\infty(\Lambda^p(\mathcal{M}))$ and $\mathbf{e_i} \in C^\infty(E)$

$$\partial v = \partial \left(\sum_{i=1}^{g} v^i \mathbf{e_i} \right) = \sum_{i=1}^{g} \left(\mathrm{d}v^i + \sum_{j=1}^{g} \varpi^i_j \wedge v^j \right) \mathbf{e_i},$$

what is usually written as

$$\partial v = \mathrm{d}v + \varpi \wedge v. \tag{5.58}$$

This defines a sequence of mappings

$$C^\infty(E) \xrightarrow{\partial} C^\infty(E \otimes \Lambda(\mathcal{M})) \xrightarrow{\partial} C^\infty(E \otimes \Lambda^2(\mathcal{M})) \xrightarrow{\partial} \cdots$$

Let us consider the second exterior covariant derivative

$$\partial^2 : C^\infty(E) \longrightarrow C^\infty(E \otimes \Lambda^2(\mathcal{M})).$$

After short computations we see that

$$\partial^2 v = \sum_{i=1}^{g} \left(\sum_{j=1}^{g} \mathrm{d}\varpi^i_j \wedge v^j + \sum_{j,k=1}^{g} \varpi^i_k \wedge \varpi^k_j \wedge v^j \right) \mathbf{e_i}$$

or in a compact form

$$\partial^2 v = (\mathrm{d}\varpi + \varpi \wedge \varpi) \wedge v. \tag{5.59}$$

Definition 5.3. *The* curvature *of the connection ϖ on the vector bundle $(E, \pi, \mathcal{M})$ is a 2-form*

$$\mathcal{R} \stackrel{\text{def.}}{=} \mathrm{d}\varpi + \varpi \wedge \varpi. \tag{5.60}$$

The transformation rule for curvature under the linear transformation (5.47) is expressed by a formula

$$\mathcal{R}' = A^{-1} \cdot \mathcal{R} \cdot A. \tag{5.61}$$

The curvature matrix

$$\mathcal{R}^i_j \stackrel{\text{def.}}{=} \mathrm{d}\varpi^i_j + \varpi^i_r \wedge \varpi^r_j = \tfrac{1}{2} R^i_{jkl}\, \mathrm{d}q^k \wedge \mathrm{d}q^l, \tag{5.62}$$

where

$$R^i_{jkl} \stackrel{\text{def.}}{=} \frac{\partial \Gamma^i_{jl}}{\partial q^k} - \frac{\partial \Gamma^i_{jk}}{\partial q^l} + \Gamma^i_{rl}\Gamma^r_{jk} - \Gamma^i_{rk}\Gamma^r_{jl} \tag{5.63}$$

is a curvature tensor of the connection determined by coefficients Γ^i_{jk}.

At the end of this introduction devoted to the general theory of connection in fibre bundles we recall procedures defining induced connections on tensor products of bundles, direct sums of bundles and a dual bundle.

Definition 5.4. *Assume that two exterior covariant derivatives (denoted by the same symbol ∂) in two vector bundles E_1 and E_2 are given. The* induced exterior covariant derivatives *on $E_1 \otimes E_2$ and $E_1 \oplus E_2$ are determined by rules*

$$\partial(v_1 \otimes v_2) = \partial v_1 \otimes v_2 + v_1 \otimes \partial v_2, \tag{5.64}$$

$$\partial(v_1 \oplus v_2) = \partial v_1 \oplus \partial v_2 \tag{5.65}$$

for $v_1 \in E_1$, $v_2 \in E_2$.

Definition 5.5. *Suppose* $v \in C^\infty(E)$, $v^* \in C^\infty(E^*)$ *and the pairing* $\langle v, v^*\rangle \in C^\infty(M)$. *The* induced exterior covariant derivative *on* E^* *is determined by the relation*

$$\mathrm{d}\langle v, v^*\rangle = \langle \partial v, v^*\rangle + \langle v, \partial v^*\rangle. \tag{5.66}$$

Relations (5.64)–(5.66) contain of course the recipe for building *induced connections* on $E_1 \otimes E_2$, $E_1 \oplus E_2$ and E^* respectively.

After that introduction we come back to the Weyl bundle and present constructions of two connections: one symplectic and one Abelian on the Weyl bundle $\mathcal{P}^*\mathcal{M}[[\hbar]]$. Both of these connections play a crucial role in the definition of the $*$-product on a symplectic manifold $\mathcal{M}$. As before we assume that $\dim \mathcal{M} = 2n$.

From the Darboux theorem (see [26]) for each point p on a symplectic manifold $\mathcal{M}$ there exists a chart $(\mathcal{U}_z, \phi_z)$ such that $\mathrm{p} \in \mathcal{U}_z$ and on $\mathcal{U}_z$ in local coordinates $(q^1, \ldots, q^{2n})$ determined by ϕ_z the symplectic form equals

$$\omega = \mathrm{d}q^{n+1} \wedge \mathrm{d}q^1 + \ldots + \mathrm{d}q^{2n} \wedge \mathrm{d}q^n.$$

A chart $(\mathcal{U}_z, \phi_z)$ is called a *Darboux chart*. A set of C^∞ compatible Darboux charts covering the whole manifold $\mathcal{A} = \{(\mathcal{U}_z, \phi_z)\}_{z\in J}$ constitutes a *Darboux atlas* on $\mathcal{M}$.

Suppose that on the tangent bundle $\mathcal{T}\mathcal{M}$ a torsion-free connection is done. This connection ϖ is locally determined by sets of smooth real functions Γ^k_{ij}, $1 \leqslant i, j, k \leqslant 2n$. Using formulae (5.66) and (5.65) we extend ϖ easily on the bundle $\mathcal{T}^*\mathcal{M} \otimes \mathcal{T}^*\mathcal{M}$.

Definition 5.6. *A torsion-free connection* ϖ *on the symplectic manifold* $\mathcal{M}$ *is called* symplectic *if at each point of* $\mathcal{M}$

$$\partial\omega = 0. \tag{5.67}$$

It can be proved [8] that each symplectic manifold may be endowed with some symplectic connection. In a Darboux chart coefficients of symplectic connection

$$\Gamma_{ijk} \stackrel{\text{def.}}{=} \omega_{li}\Gamma^l_{jk} \tag{5.68}$$

are symmetric in all the indices $\{i, j, k\}$. The matrix of symplectic connection

$$\varpi^i_j = \sum_{k=1}^{2n} \Gamma^i_{jk}\, \mathrm{d}q^k.$$

A symplectic manifold equipped with a symplectic connection is often called a Fedosov manifold.

Let us consider the collection

$$(\mathcal{T}^*\mathcal{M})^l \stackrel{\text{def.}}{=} \bigcup_{\mathrm{p}\in\mathcal{M}} (T^*_{\mathrm{p}}M)^l \tag{5.69}$$

of spaces $(T^*_{\mathrm{p}}M)^l$ taken at all points of the symplectic manifold $\mathcal{M}$. It is easy to note that this collection is a vector bundle which we will denote by $((\mathcal{T}^*\mathcal{M})^l, \pi, \mathcal{M})$. For $l \geqslant 1$ we are able to introduce a local frame field on $((\mathcal{T}^*\mathcal{M})^l, \pi, \mathcal{M})$

$$\tilde{\mathbf{e}}_1 \stackrel{\text{def.}}{=} \underbrace{\mathrm{d}q^1 \odot \mathrm{d}q^1 \odot \cdots \odot \mathrm{d}q^1}_{\text{l-times}},$$

$$\tilde{\mathbf{e}}_2 \stackrel{\text{def.}}{=} \underbrace{\mathrm{d}q^1 \odot \mathrm{d}q^1 \odot \ldots \odot \mathrm{d}q^2}_{\text{l-times}},$$

$$\vdots$$

$$\tilde{\mathbf{e}}_{\frac{(2n+l-1)!}{l!(2n-1)!}} \stackrel{\text{def.}}{=} \underbrace{\mathrm{d}q^{2n} \odot \mathrm{d}q^{2n} \odot \cdots \odot \mathrm{d}q^{2n}}_{\text{l-times}}.$$

The relation between indices $i_1, \ldots, i_l$ in the symmetric tensor product $\tilde{\mathbf{e}}_{\mathbf{k}} = \mathrm{d}q^{i_1} \odot \mathrm{d}q^{i_2} \odot \cdots \odot \mathrm{d}q^{i_l}$ and the number k is given by the formula

$$k = \binom{2n+l-1}{l} - \sum_{s=1}^{l} \binom{2n+s-i_{l-s+1}-1}{s}. \tag{5.70}$$

The proof of this fact is given in the appendix.

We look for the matrix of the induced connection on $(\mathcal{T}^*\mathcal{M})^l$. From (5.64)

$$\partial(\mathrm{d}q^{i_1} \odot \mathrm{d}q^{i_2} \odot \cdots \odot \mathrm{d}q^{i_l}) = \sum_{j=1}^{2n} \sum_{r=1}^{l} -\varpi^j_{i_r} \mathrm{d}q^j \odot \mathrm{d}q^{i_1} \odot \cdots \odot \mathrm{d}\check{q}^{i_r} \odot \cdots \odot \mathrm{d}q^{i_l}.$$

As usual the symbol $\mathrm{d}\check{q}^{i_r}$ denotes the omitted element.

Let us consider the kth row of a matrix ${}_l\varpi$ representing the induced connection on $(\mathcal{T}^*\mathcal{M})^l$. We conclude that different from 0 may be only terms ${}_l\varpi^m_k$ such that among l parameters i_r determining (via (5.70)) number m at least $(l-1)$ have been taken from the set $i_1, \ldots, i_l$. Matrix ${}_l\varpi$ is $\frac{(2n+l-1)!}{l!(2n-1)!} \times \frac{(2n+l-1)!}{l!(2n-1)!}$ dimensional.

The example. Given a connection matrix ϖ on the tangent bundle $\mathcal{TM}$ assuming that $n = 1, l = 2$. The local frame system of the bundle $(\mathcal{T}^*\mathcal{M})^2$ contains three elements: $\mathrm{d}q^1 \odot \mathrm{d}q^1, \mathrm{d}q^1 \odot \mathrm{d}q^2$ and $\mathrm{d}q^2 \odot \mathrm{d}q^2$. The matrix of induced connection ${}_2\varpi$ on $(\mathcal{T}^*\mathcal{M})^2$ looks like

$${}_2\varpi = \begin{bmatrix} -2\varpi^1_1 & -2\varpi^2_1 & 0 \\ -\varpi^1_2 & -\varpi^1_1 - \varpi^2_2 & -\varpi^2_1 \\ 0 & -2\varpi^2_1 & -2\varpi^2_2 \end{bmatrix}.$$

In section 3 analysing the construction of the Weyl vector space we showed that the order of elements in formula (3.21) is determined by the degree, the power of $\hbar$, indices of tensors and finally the real or imaginary nature of the element. Using these facts we represent the Weyl algebra bundle as a double direct sum

$$\mathcal{P}^*\mathcal{M}[[\hbar]] = \bigoplus_{z=0}^{\infty} \bigoplus_{k=0}^{[\frac{z}{2}]} ((\mathcal{T}^*\mathcal{M})^{z-2k} \oplus (\mathcal{T}^*\mathcal{M})^{z-2k}). \tag{5.71}$$

The local frame field of the Weyl bundle $\mathcal{P}^*\mathcal{M}[[\hbar]]$ on $\mathcal{U} \subset \mathcal{M}$ contains

$$\mathbf{e_1} \stackrel{\text{def.}}{=} \hat{\imath} \oplus \theta \oplus \theta \oplus \cdots,$$
$$\mathbf{e_2} \stackrel{\text{def.}}{=} \theta \oplus \hat{\imath} \oplus \theta \oplus \cdots,$$
$$\mathbf{e_3} \stackrel{\text{def.}}{=} \theta \oplus \theta \oplus \mathrm{d}q^1 \oplus \theta \oplus \cdots,$$
$$\mathbf{e_4} \stackrel{\text{def.}}{=} \theta \oplus \theta \oplus \theta \oplus \mathrm{d}q^1 \oplus \cdots,$$
$$\vdots$$

The relation between an index a in $\mathbf{e_a}$ and indices $k, i_i, \ldots, i_l$ is expressed by formulae (A.89) and (A.90) from the appendix. By $\hat{\imath}$ we understand a versor with unitary length on a real line $\mathbb{R}$. The letter θ denotes the neutral element in the vector space $\mathbb{R}$.

In the local frame field for a fixed even z the matrix of induced connection

$$\overset{\mathbf{z}}{\varpi} = \begin{bmatrix} {}_z\varpi & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & {}_z\varpi & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & {}_1\varpi & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

or, for odd z

$$\overset{\mathbf{z}}{\varpi} = \begin{bmatrix} {}_z\varpi & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & {}_z\varpi & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & {}_1\varpi \end{bmatrix}.$$

Each matrix $\overset{\mathbf{z}}{\varpi}$ is a square even-dimensional matrix.

The matrix of induced connection on the Weyl bundle

$$\varpi = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \overset{\mathbf{1}}{\varpi} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{0} & \overset{\mathbf{2}}{\varpi} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{5.72}$$

In Darboux coordinates coefficients of the symplectic connection (5.68) are symmetric in all indices, so we may express the exterior covariant derivative in terms of the $\circ$-product. Indeed, let us introduce a connection 1-form as an element of $C^\infty((\mathcal{T}^*\mathcal{M})^2 \otimes \Lambda^1(\mathcal{M}))$

$$\Gamma = \Gamma[0,2]_{ij,k}\,\mathrm{d}q^k \stackrel{\text{def.}}{=} \Gamma_{ij,k}\,\mathrm{d}q^k.$$

Moreover,

Definition 5.7. *The commutator of two smooth sections $u \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{j_1}(\mathcal{M}))$ and $v \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{j_2}(\mathcal{M}))$ is a smooth section of $C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{j_1+j_2}(\mathcal{M}))$ such that*

$$[u,v] \stackrel{\text{def.}}{=} u \circ v - (-1)^{j_1 \cdot j_2} v \circ u. \tag{5.73}$$

In further considerations we simplify a bit the notation by putting $\mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda(\mathcal{M})$ instead of $\sum_{j=1}^{2n} \mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^j(\mathcal{M})$.

Theorem 5.2. *In Darboux coordinates the exterior covariant derivative ∂v of $v \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda(\mathcal{M}))$ reads*

$$\partial v = \mathrm{d}v + \frac{1}{\mathrm{i}\hbar}[\Gamma, v]. \tag{5.74}$$

The reason why we restrict ourselves to Darboux charts is obvious —only in such charts is the connection represented by a form with values in the Weyl algebra.

In the Fedosov paper [6] the above relation is just a definition of the exterior covariant derivative in the Weyl bundle. We showed its geometrical origin.

From the fact that the exterior covariant derivative in the Weyl bundle can be put into frames of the algebra structure, we deduce that also curvature of the symplectic connection may be defined in terms of $\mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda(\mathcal{M})$. Indeed, curvature of the symplectic connection Γ is a smooth section $C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^2(\mathcal{M}))$ and can be written as

$$\mathcal{R} = \mathrm{d}\Gamma + \frac{1}{\mathrm{i}\hbar}\Gamma \circ \Gamma. \tag{5.75}$$

The action of $\mathcal{R}$ on a smooth section X of a tangent bundle $\mathcal{TM}$ is given by

$$\mathcal{R}(X) = \tfrac{1}{4} R_{ijkl} X^i X^j \,\mathrm{d}q^k \wedge \mathrm{d}q^l,$$

where

$$R_{ijkl} = \frac{\partial \Gamma_{ijl}}{\partial q^k} - \frac{\partial \Gamma_{ijk}}{\partial q^l} + \omega^{zu}\Gamma_{zil}\Gamma_{ujk} - \omega^{zu}\Gamma_{zik}\Gamma_{ujl}. \tag{5.76}$$

Thus the second exterior covariant derivative of $v \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda(\mathcal{M}))$ is described by a formula

$$\partial^2 v = \frac{1}{\mathrm{i}\hbar}[\mathcal{R}, v]. \tag{5.77}$$

6. The $*$-product on a symplectic manifold $\mathcal{M}$

The crucial role in deformation quantization is played by a noncommutative nonAbelian $*$-product being a counterpart of the product of operators from Hilbert space formulation of quantum theory. There is no unique way to introduce this product in the case when the phase space of the system is different from $\mathbb{R}^{2n}$. In this section we present briefly the Fedosov construction of the $*$-product on a symplectic manifold $(\mathcal{M}, \omega)$. We concentrate on a geometric aspect of the problem omitting all technical proofs.

There exists infinitely many different connections in the bundle $\mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda(\mathcal{M})$. Especially important to our purposes is one of the so-called Abelian connections $\tilde{\Gamma}$.

Definition 6.8. *A connection $\tilde{\Gamma}$ is called* Abelian*, if its curvature Ω is a 2-form with values in the bundle $\bigoplus_{k=0}^{\infty}\left((\mathcal{T}^*\mathcal{M})_k^0 \oplus (\mathcal{T}^*\mathcal{M})_k^0\right) \otimes \Lambda^2(\mathcal{M})$.*

In the definition we put $(\mathcal{T}^*\mathcal{M})_k^0 = (\mathcal{T}^*\mathcal{M})^0$ for every k.

Since that for each $v \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda(\mathcal{M}))$ the second exterior covariant derivative D determined by $\tilde{\Gamma}$ equals (compare (5.77))

$$D^2 v = \frac{1}{\mathrm{i}\hbar}[\Omega, v] = 0. \tag{6.78}$$

From among the set of Abelian connections especially important for physical applications is that of the form

$$\tilde{\Gamma}(X) \stackrel{\text{def.}}{=} \omega_{i,j} X^i \,\mathrm{d}q^j + \Gamma_{i_1 i_2, j} X^{i_1} X^{i_2} \,\mathrm{d}q^j + r. \tag{6.79}$$

The term $r \in \mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^1(\mathcal{M})$, $\deg(r) \geqslant 3$ is determined by some recurrential formula (for details see [6, 7]) and it depends only on the symplectic curvature $\mathcal{R}$. In this case

$$\Omega = -\tfrac{1}{2}\omega_{j_1 j_2}\,\mathrm{d}q^{j_1} \wedge \mathrm{d}q^{j_2}.$$

To avoid confusion which indices are of the Weyl bundle and which are of a differential form we put in (6.79) the result of acting by $\tilde{\Gamma}$ on some arbitrary fixed vector field from the tangent bundle $\mathcal{TM}$.

It is easy to note that the component of the Abelian connection (6.79) with the degree z acting on the form $v \in \mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda(\mathcal{M})$ gives as a result the element $u \in \mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda(\mathcal{M})$ for which

$$\deg(u) = \deg(v) + z - 2.$$

Since denoting by $\tilde{\varpi}^{(\deg(v))}_{(\deg(u))}$ a part of the connection matrix ϖ appearing as the result of the presence of the Abelian connection $\tilde{\Gamma}$ we see that

$$\tilde{\varpi} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ \tilde{\varpi}^{(2)}_{(1)} & \overset{\mathbf{1}}{\varpi} & \tilde{\varpi}^{(2)}_{(3)} & \dots \\ \mathbf{0} & \tilde{\varpi}^{(3)}_{(2)} & \overset{\mathbf{2}}{\varpi} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{6.80}$$

(compare (5.72)). Note that the Abelian connection $\tilde{\Gamma}$ mixes up tensors of different ranges and terms standing at different powers of the deformation parameter $\hbar$.

What is interesting, the set of 0-forms belonging to $\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^0(\mathcal{M})$, such that $Dv = 0$, constitutes the subalgebra of the Weyl algebra $\mathcal{P}^*\mathcal{M}[[\hbar]]$. We will denote this subalgebra by $\mathcal{P}^*\mathcal{M}_D[[\hbar]]$.

Definition 6.9. *A* projection

$$\sigma : \mathcal{P}^*\mathcal{M}_D[[\hbar]] \otimes \Lambda^0(\mathcal{M}) \longrightarrow \bigoplus_{k=0}^{\infty} \left((\mathcal{T}^*\mathcal{M})^0_k \oplus (\mathcal{T}^*\mathcal{M})^0_k\right)$$

assigns to each section $v \in \mathcal{P}^*\mathcal{M}_D[[\hbar]] \otimes \Lambda^0(\mathcal{M})$ *its part* $\sigma(v) \in \bigoplus_{k=0}^{\infty}\left((\mathcal{T}^*\mathcal{M})^0_k \oplus (\mathcal{T}^*\mathcal{M})^0_k\right)$.

It has been proved (see [6, 7]) that for each $f \in C^\infty\left(\bigoplus_{k=0}^{\infty}((\mathcal{T}^*\mathcal{M})^0_k \oplus (\mathcal{T}^*\mathcal{M})^0_k)\right)$ there exists a *unique* smooth section $v \in C^\infty(\mathcal{P}^*\mathcal{M}_D[[\hbar]] \otimes \Lambda^0(\mathcal{M}))$ such that $\sigma(v) = f$. We arrive at the point crucial for physical applications of the mathematical machinery presented above.

Definition 6.10. *Let* f_1, f_2 *be two smooth sections of* $C^\infty\left(\bigoplus_{k=0}^{\infty}\left((\mathcal{T}^*\mathcal{M})^0_k \oplus (\mathcal{T}^*\mathcal{M})^0_k\right)\right)$. *The* $*$*-product of them is defined as*

$$f_1 * f_2 \stackrel{\text{def.}}{=} \sigma\left(\sigma^{-1}(f_1) \circ \sigma^{-1}(f_2)\right). \tag{6.81}$$

This $*$-product defined here can be considered as a generalization of the Moyal product of the Weyl type defined for $M = \mathbb{R}^{2n}$. It has the following properties:

1. The definition of $*$-product is invariant under Darboux transformations. Hence we can multiply functions on an arbitrary symplectic manifold. In the case of the Moyal product of the Weyl type we are restricted to the symplectic manifold $\mathbb{R}^{2n}$ and Darboux coordinates with vanishing 1-form of a symplectic connection $\Gamma = 0$. In fact we could extend the definition of $*$-product on an arbitrary atlas on the symplectic manifold $\mathcal{M}$, but in such a case it would be necessary to modify recurrential formulae for the term r in (6.79) and for σ^{-1}.
2. In the limit $\hbar \to 0^+$ the $*$-product of $f_1, f_2 \in C^\infty\left((\mathcal{T}^*\mathcal{M})^0_0 \oplus (\mathcal{T}^*\mathcal{M})^0_0\right)$ turns into the commutative point-wise multiplication of functions, i.e.

$$\lim_{\hbar\to 0^+} f_1 * f_2 = f_1 \cdot f_2. \tag{6.82}$$

   This relation expresses the fact that the classical mechanics is a limit of quantum physics for the Planck constant tending to 0^+.

3. Multiplication (6.81) is associative but noncommutative. The noncommutativity of two smooth sections $f_1, f_2 \in C^\infty\left(\bigoplus_{k=0}^{\infty}\left((\mathcal{T}^*\mathcal{M})^0_k \oplus (\mathcal{T}^*\mathcal{M})^0_k)\right)\right)$ is measured by their Moyal bracket
$$\{f_1, f_2\}_M \stackrel{\text{def.}}{=} \frac{1}{\mathrm{i}\hbar}(f_1 * f_2 - f_2 * f_1)$$
which plays a role of commutator from standard formulation of quantum mechanics on Hilbert space.
4. When $\mathcal{M} = \mathbb{R}^{2n}$ the product defined above is just the Moyal product of Weyl type. This fact confirms consistency of the Fedosov approach with the best known case of quantum deformation.

Some computations done with the $*$-product (6.81) can be found in [8].

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Appendix. Position of the element $v[k, l]_{i_1\ldots i_l}$ in the series (3.21)

In this appendix we shall explain the relation between indices $k, i_1, \ldots i_l$ of an element $v[k, l]_{i_1\ldots i_l}$ of the Weyl algebra and its position in the series
$$\phi(v) = (\mathrm{Re}(v[0,0]), \mathrm{Im}(v[0,0]), \mathrm{Re}(v[0,1]_1), \mathrm{Im}(v[0,1]_1), \ldots)$$
representing a vector v of the Weyl space $P^*_{\mathrm{p}}M[[\hbar]]$. Formulae proved in this appendix are also useful in further considerations on the construction of a local frame field in the Weyl bundle.

We will achieve our aim in four steps.

1. At the beginning we find the place P_1 of $v[k, l]_{i_1\ldots i_l}$ among elements with the same k and l.
2. After that we consider the relation between a power k of $\hbar$ and the position P_2 for elements with the same degree $2k + l$.
3. In the third step we include the influence of the degree on the position P_3.
4. Finally we present the complete formula expressing the relation between the real and imaginary part of $v[k, l]_{i_1\ldots i_l}$ and its position $\mathbf{P}$ in (3.21).

Let us start from the case when k is fixed and $l \geqslant 1$. We look for the position of $v[k, l]_{i_1\ldots i_l}$ in the series of real elements parametrizing by the same k and l. Elements are ordered according to the rule valid for relation (3.10). The position $P_1(v[k, l]_{i_1\ldots i_l})$ of $v[k, l]_{i_1\ldots i_l}$ in that series equals

$$\sum_{z_1=1}^{i_1-1}\sum_{z_2=z_1}^{2n}\cdots\sum_{z_l=z_{l-1}}^{2n} 1 + \sum_{z_2=i_1}^{i_2-1}\sum_{z_3=z_2}^{2n}\cdots\sum_{z_l=z_{l-1}}^{2n} 1 + \cdots + \sum_{z_{l-1}=i_{l-2}}^{i_{l-1}-1}\sum_{z_l=z_{l-1}}^{2n} 1 + \sum_{z_l=i_{l-1}}^{i_l} 1$$
$$= \sum_{z_1=1}^{i_1-1}\sum_{z_2=z_1}^{2n}\cdots\sum_{z_l=z_{l-1}}^{2n} 1 + \sum_{z_2=1}^{i_2-i_1}\sum_{z_3=z_2}^{2n-i_1+1}\cdots\sum_{z_l=z_{l-1}}^{2n-i_1+1} 1$$
$$+ \cdots + \sum_{z_{l-1}=1}^{i_{l-1}-i_{l-2}}\sum_{z_l=z_{l-1}}^{2n-i_{l-2}+1} 1 + \sum_{z_l=1}^{i_l-i_{l-1}+1} 1. \tag{A.83}$$

Let us consider the following expression:

$$\sum_{z_1=1}^{m}\sum_{z_2=z_1}^{m}\cdots\sum_{z_l=z_{l-1}}^{m} 1 = \sum_{z_1=1}^{a}\sum_{z_2=z_1}^{m}\cdots\sum_{z_l=z_{l-1}}^{m} 1 + \sum_{z_1=a+1}^{m}\sum_{z_2=z_1}^{m}\cdots\sum_{z_l=z_{l-1}}^{m} 1$$
$$= \sum_{z_1=1}^{a}\sum_{z_2=z_1}^{m}\cdots\sum_{z_l=z_{l-1}}^{m} 1 + \sum_{z_1=1}^{m-a}\sum_{z_2=z_1}^{m-a}\cdots\sum_{z_l=z_{l-1}}^{m-a} 1$$
$$\stackrel{(3.11)}{=} \sum_{z_1=1}^{a}\sum_{z_2=z_1}^{m}\cdots\sum_{z_l=z_{l-1}}^{m} 1 + \frac{(m+l-a-1)!}{l!(m-a-1)!}.$$

But from (3.11) we know that $\sum_{z_1=1}^{m}\sum_{z_2=z_1}^{m}\cdots\sum_{z_l=z_{l-1}}^{m} 1 = \frac{(m+l-1)!}{l!(m-1)!}$ so

$$\sum_{z_1=1}^{a}\sum_{z_2=z_1}^{m}\cdots\sum_{z_l=z_{l-1}}^{m} 1 = \frac{(m+l-1)!}{l!(m-1)!} - \frac{(m+l-a-1)!}{l!(m-a-1)!}. \tag{A.84}$$

Relation (A.84) works also for $l = 1$. In this case the right-hand side of (A.84) is independent of m and we see that $\sum_{z_1=1}^{a} = a$. Substituting (A.84) into (A.83) we have

$$P_1(v[k,l]_{i_1\ldots i_l}) = \frac{(2n+l-1)!}{l!(2n-1)!} - \frac{(2n+l-i_1)!}{l!(2n-i_1)!} + \frac{(2n+l-i_1-1)!}{(l-1)!(2n-i_1)!}$$
$$- \frac{(2n+l-i_2-1)!}{(l-1)!(2n-i_2)!} + \cdots + \frac{(2n-i_{l-2}+2)!}{2!(2n-i_{l-2})!}$$
$$- \frac{(2n-i_{l-1}+2)!}{2!(2n-i_{l-1})!} + (i_l - i_{l-1} + 1).$$

Putting together terms containing i_{l_r} we see that

$$P_1(v[k,l]_{i_1\ldots i_l}) = \frac{(2n+l-1)!}{l!(2n-1)!} - \frac{(2n+l-i_1-1)!}{l!(2n-i_1-1)!} - \cdots - \frac{(2n-i_{l-1}+1)!}{2!(2n-i_{l-1}-1)!}$$
$$- \frac{(2n-i_l)!}{1!(2n-i_l-1)!}.$$

Note that although

$$-\frac{(2n-i_{l-1}+1)!}{2!(2n-i_{l-1}-1)!} = -\frac{(2n-i_{l-1}+2)!}{2!(2n-i_{l-1})!} - i_{l-1} + 1 + \mathbf{2n}$$

and

$$-\frac{(2n-i_l)!}{1!(2n-i_l-1)!} = i_l - \mathbf{2n}$$

the sum

$$-\frac{(2n-i_{l-1}+1)!}{2!(2n-i_{l-1}-1)!} - \frac{(2n-i_l)!}{1!(2n-i_l-1)!} = -\frac{(2n-i_{l-1}+2)!}{2!(2n-i_{l-1})!} - i_{l-1} + 1 + i_l$$

as is required.

Finally we may write

$$P_1\big(v[k,l]_{i_1\ldots i_l}\big) = \binom{2n+l-1}{l} - \sum_{s=1}^{l}\binom{2n+s-i_{l-s+1}-1}{s}. \tag{A.85}$$

Note that the above relation is true also for scalars, i.e. when $l = 0$.

In the next step we take into account the degree of $v[k,l]_{i_1\ldots i_l}$. From definition 3.5 we see that $\deg\big(v[k,l]_{i_1\ldots i_l}\big) = 2k+l$. We denote it by d. The position $P_2\big(v[k,l]_{i_1\ldots i_l}\big)$ of $v[k,l]_{i_1\ldots i_l}$ among elements with the same degree is given by

$$\begin{aligned} P_2(v[k,l]_{i_1\ldots i_l}) &= \sum_{g=0}^{k-1}\binom{2n+d-2g-1}{d-2g} + P_1\big(v[k,l]_{i_1\ldots i_l}\big) \\ &= -\binom{2n+l-1}{l}\,{}_3F_2\left(\left\{1,\frac{1}{2}-\frac{l}{2},-\frac{l}{2}\right\},\left\{\frac{1}{2}-\frac{l}{2}-n,1-\frac{l}{2}-n\right\},1\right) \\ &\quad+\binom{2n+2k+l-1}{2k+l}\,{}_3F_2\left(\left\{1,-k-\frac{l}{2},\frac{1}{2}-\frac{l}{2}-k\right\},\right. \\ &\quad\left.\left\{\frac{1}{2}-\frac{l}{2}-k-n,1-\frac{l}{2}-k-n\right\},1\right)+P_1\big(v[k,l]_{i_1\ldots i_l}\big). \end{aligned} \tag{A.86}$$

By ${}_3F_2(\{a_1,a_2,a_3\},\{b_1,b_2\},x)$ we denote the generalized hypergeometric function (see [29]). From (A.86) we obtain that the total number of real elements of the same degree d is

$$\begin{aligned} \sum_{g=0}^{[\frac{d}{2}]}\binom{2n+d-2g-1}{d-2g} &= \binom{2n+d-1}{d} \\ &\quad\times{}_3F_2\left(\left\{1,\frac{1}{2}-\frac{d}{2},-\frac{d}{2}\right\},\left\{\frac{1}{2}-\frac{d}{2}-n,1-\frac{d}{2}-n\right\},1\right). \end{aligned} \tag{A.87}$$

Since we conclude that

$$\begin{aligned} &P_3(v[k,l]_{i_1\ldots i_l}) = P_2(v[k,l]_{i_1\ldots i_l}) \\ &+\sum_{c=0}^{d-1}\binom{2n+c-1}{c}\,{}_3F_2\left(\left\{1,\frac{1}{2}-\frac{c}{2},-\frac{c}{2}\right\},\left\{\frac{1}{2}-\frac{c}{2}-n,1-\frac{c}{2}-n\right\},1\right) \\ &= P_2(v[k,l]_{i_1\ldots i_l}) + \begin{cases} -\frac{1}{4^{n+1}}-\binom{2n+d+1}{d+1}\,{}_3F_2\left(\left\{1,1+\frac{d}{2}+n,\frac{3}{2}+\frac{d}{2}+n\right\},\right. \\ \left.\left\{1+\frac{d}{2},\frac{3}{2}+\frac{d}{2}\right\},1\right) \qquad \text{for even } d \\ -\frac{1}{4^{n+1}}-\binom{2n+d}{d}\,{}_3F_2\left(\left\{1,\frac{1}{2}+\frac{d}{2}+n,1+\frac{d}{2}+n\right\},\right. \\ \left.\left\{\frac{1}{2}+\frac{d}{2},1+\frac{d}{2}\right\},1\right) \\ +\binom{2n+d-2}{d-1}\,{}_3F_2\left(\left\{1,\frac{1}{2}-\frac{d}{2},1-\frac{d}{2}\right\},\right. \\ \left.\left\{1-\frac{d}{2}-n,\frac{3}{2}-\frac{d}{2}-n\right\},1\right) \qquad \text{for odd } d. \end{cases} \end{aligned} \tag{A.88}$$

Finally we arrive at the result that the position of the element $v[k,l]_{i_1\ldots i_l}$ in the series (3.21) is described by two relations

$$\mathbf{P}\big(\mathrm{Re}(v[k,l]_{i_1\ldots i_l})\big) = 2P_3\big(v[k,l]_{i_1\ldots i_l}\big)-1 \tag{A.89}$$

and

$$\mathbf{P}\big(\mathrm{Im}\big(v[k,l]_{i_1\ldots i_l}\big)\big) = 2P_3\big(v[k,l]_{i_1\ldots i_l}\big). \tag{A.90}$$

Mappings (A.89) and (A.90) are one-to-one so from the position of the element in series (3.21) it is possible to reconstruct the sequence $k, i_1, \ldots, i_l$.

References

[1] Moyal J E 1949 *Proc. Camb. Phil. Soc.* **45** 99

[2] Weyl H 1931 *The Theory of Groups and Quantum Mechanics* (New York: Dover)

[3] Wigner E P 1932 *Phys. Rev.* **40** 749

[4] Groenewold H J 1946 *Physica* **12** 405

[5] Bayen F, Flato M, Fransdal C, Lichnerowicz A and Sternheimer D 1978 *Ann. Phys., NY* **111** 61
Bayen F, Flato M, Fransdal C, Lichnerowicz A and Sternheimer D 1978 *Ann. Phys., NY* **111** 111

[6] Fedosov B 1994 *J. Diff. Geom.* **40** 213

[7] Fedosov B 1996 *Deformation Quantization and Index Theory* (Berlin: Akademie Verlag)

[8] Gadella M, del Olmo M A and Tosiek J Geometrical origin of the ∗-product in the Fedosov formalism *J. Geom. Phys.* at press

[9] Lange S 1995 *Differential and Riemannian Geometry* (New York: Springer)

[10] Asakawa T and Kishimoto I 2000 *Preprint* hep-th/0002138

[11] Formański S and Przanowski M 2004 *Preprint* math-ph/0412011

[12] Plymen R 1982 *J. Funct. Anal.* **49** 186

[13] Bordemann M and Waldmann S 1997 *Lett. Math. Phys.* **41** 243

[14] Bordemann M, Neumaier N and Waldmann S 1997 *Preprint* q-alg/9711016

[15] Bordemann M 1996 *Preprint* q-alg/9605038

[16] Bordemann M, Herbig H and Waldmann S 1999 *Preprint* mathQA/9901015

[17] Bordemann M, Neumaier N, Pflaum M J and Waldmann S 1998 *Preprint* math/9811055

[18] Bordemann M and Waldmann S 1998 *Commun. Math. Phys.* **195** 549

[19] Przanowski M and Tosiek J 1999 *Acta Phys. Pol.* B **30** 179

[20] Maurin K 1980 *Integration, Distributions, Holomorphic Functions, Tensor and Harmonic Analysis* (London: Reidel)

[21] Kuratowski K 1966 *Topology* (New York: Academic)

[22] Jänich K 1984 *Topology* (Berlin: Springer)

[23] Nakahara M 1990 *Geometry, Topology and Physics* (Bristol: Hilger)

[24] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* (New York: Wiley)

[25] Gadella M, del Olmo M A and Tosiek J 2003 *Ann. Phys., NY* **307** 272

[26] Cannas da Silva A 2001 *Lectures on Symplectic Geometry* (Berlin: Springer)

[27] Chern S S, Chen W H and Lam K S 1999 *Lectures on Differential Geometry* (Singapore: World Scientific)

[28] Ward R S and Wells R O Jr 1989 *Twistor Geometry and Field Theory* (Cambridge: Cambridge University Press)

[29] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series, and Products* (New York: Academic)